Mathematics Methods

Unit 3 & 4

Integration

1.	**Indefinite integration rules**
	(a) Increase the power by one and divide by the new power
	$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$
	Example: Integrate $f'(x) = 2x$.
	(b) Others

By substitution	**By formula**
$$\int (ax+b)^n\,dx$$	$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq 1$$
Example: $\int (5-3x)^2\ dx$	

Trigonometric functions

$$\int cos\ x\ dx = sin\ x + c$$
$$\int cos\ ax\ dx = \frac{1}{a} sin\ x + c$$
$$\int sin\ x\ dx = -cos\ x + c$$
$$\int sin\ ax\ dx = -\frac{1}{a} cos\ x + c$$

$$\int sec^2 x \ dx = tan\, x + c$$

Example 1:
Integrate $15\, cos\, 5x$.

Example 2:
Integrate $sin\, 5x + 6x$

Example 3:
Integrate $cos\, 5x\, cos\, 5x - sin\, 5x\, sin\, 5x$.

Exponential functions

$$\int e^x \ dx = e^x + c$$

Example 1:
Integrate e^{2x}.

Example 2:
Integrate $5e^{3x} + 3x$.

Example 3:
Integrate $6e^{3x+1}$.

Logarithmic functions

$$\int \frac{1}{x}\ dx = \ln|x| + c$$
$$\int \frac{1}{ax+b}\ dx = ln(ax+b) + c$$

Example 1:
Integrate $\frac{7}{x}$.

Example 2:
Integrate $\frac{1}{6x}$.

Example 3:
Integrate $\frac{1}{4x+5}$.

Example 4:
Integrate $\frac{4x}{4x^2+5}$.

Example 5:
Integrate $x + \frac{1}{x}$.

Example 6:
Integrate $tan\, 2\theta$.

Example 7:
Integrate $\frac{cos\, x}{sin\, x} + \frac{1}{x}$.

2. Integration involving partial fraction

Cases for setting up a partial fraction

Case	Rational function	Partial fraction
Distinct linear factors	$\dfrac{px+q}{(x-a)(x-b)}, a \neq b$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$
Distinct cubic linear factors	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$
Repeated linear factors	$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
	$\dfrac{px+q}{(x-a)^3}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3}$
Repeated linear and distinct linear factors	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$

Example 1:

Find the values of A, B and C given that $\frac{x^2+11}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$. Hence, evaluate $\int_1^2 \frac{x^2+11}{(x+2)^2(x-3)}\,dx$.

Example 2:
Find the values of A and B given that $\frac{3-x}{5+3x-2x^2} = \frac{A}{5-2x} + \frac{B}{1+x}$. Hence evaluate $\int_0^2 \frac{3-x}{5+3x-2x^2}\ dx$.

3. The arbitrary constant, "c" in indefinite integration

(a) Origin

Origin of arbitrary constant (by example):
By differentiating $y = mx + c$, we can get $\frac{dy}{dx} = m$. The value of c disappears as it does not have an unknown, x.

$$y = 3x + 1$$
$$y = 3x + 2$$
$$y = 3x + 3$$
$$.....$$

For all the equations above,

$$\frac{dy}{dx} = 3$$

If we integrate $\frac{dy}{dx} = 3$,

$$\int 3\,dx = 3x$$

From here, we can see that the equation is $y = 3x$. However, there should be a constant as

$$y = 3x + 1 \quad \neq \quad y = 3x + 2 \quad \neq \quad y = 3x + 3$$

Therefore, the integration of these equations should give $y = 3x + c$ where c is a constant, $c = 1,2,3$ for this case.

(b) How different ways of integration affects arbitrary constant

Example 1:

Method 1

$$\int cos^3x\ sin\ x\ dx = \int cos\ x\ cos^2x\ sin\ x\ \ dx$$
$$= \int (1 - sin^2x)\ cos\ x\ sin\ x\ dx$$

Let $u = sin\ x$,

$$\frac{du}{dx} = cos\ x$$

$$\int (1 - u^2)\ u\ cos\ x\ \frac{du}{cos\ x}$$
$$= \int u - u^3\ \ du$$
$$= \frac{u^2}{2} - \frac{u^4}{4} + c$$
$$= \frac{sin^2x}{2} - \frac{sin^4x}{4} + c$$
$$= \frac{1 - cos^2x}{2} - \frac{(1 - cos^2x)^2}{4} + c$$
$$= \frac{1 - cos^2x}{2} - \frac{1 - 2cos^2x + cos^4x}{4} + c$$
$$= \frac{1}{2} - \frac{cos^2x}{2} - \frac{1}{4} + \frac{cos^2x}{4} - \frac{cos^4x}{4} + c$$
$$= -\frac{cos^4x}{4} + \frac{1}{4} + c$$

Method 2

$$\int cos^3x\ sin\ x\ dx$$

Let $u = cos\ x$,

$$\frac{du}{dx} = -sin\ x$$

$$\int u^3\ sin\ x\ -\frac{du}{sin\ x}$$
$$= \int -\frac{u^4}{4}\ du$$
$$= -\frac{cos^4x}{4} + C$$

Both answers are correct. By comparing both answers,

$$C = \frac{1}{4} + c$$

Or also

$$c = C - \frac{1}{4}$$

Example 2:

Method 1

$$\int x+1\,dx = \frac{x^2}{2} + x + c$$

Method 2

$$\int x+1\,dx$$

Let $u = x + 1$

$$\int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(x+1)^2}{2} + C$$

$$= \frac{x^2}{2} + x + \frac{1}{2} + C$$

Both answers are correct. By comparing both answers,

$$c = \frac{1}{2} + C$$

Or also

$$C = c - \frac{1}{2}$$

Example 3:

Method 1

$$\int \frac{7}{5x}\,dx = \frac{7}{5}\int \frac{1}{x}\,dx$$

$$= \frac{7}{5}\ln x + c$$

Method 2

$$\int \frac{7}{5x}\,dx = \frac{7}{5}\int \frac{5}{5x}\,dx$$

$$= \frac{7}{5}\ln 5x + C$$

$$= \frac{7}{5}\ln 5 + \frac{7}{5}\ln x + C$$

$$= \frac{7}{5}\ln x + \frac{7}{5}\ln 5 + C$$

Both answers are correct. By comparing both answers,

$$c = \frac{7}{5}\ln 5 + C$$

Or also

$$C = c - \frac{7}{5}\ln 5$$

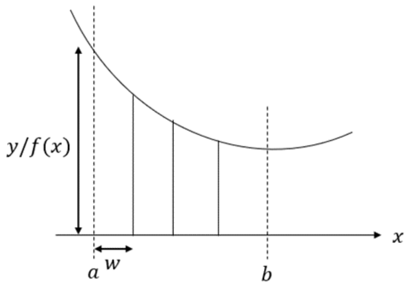
4.	**Finding equation of a curve**
	Example 1: Find the equation of curve passing with gradient function $f'(x) = 5x^2 + 2x$ at $(3,5)$. Example 2: Find v given that $\frac{dv}{dt} = \frac{50t}{(t^2-1)^2}$ at $(2,3)$. Example 3: The tangent to the curve $y = f(x)$ at point $(2,0)$ is equated by $y = 2x + 3$. The gradient function is $f'(x) = zx + h$. What is the equation of curve that it passes through $(4,7)$?

5.	**Area under the curve**

(a) Trapezium rule

Additional info.

Given a curve with function $f(x)$



To find each area of strips (trapezium):

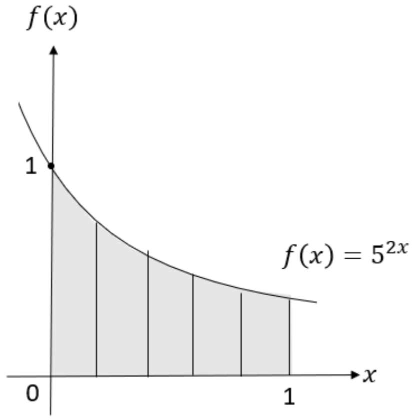
$$Area = \frac{1}{2}(y_0 + y_1)\, w$$

Total area under the curve by calculating the total area of rectangular strips,

$$Area\ = w\left[\frac{1}{2}(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots y_{n-1})\right]$$

Example:

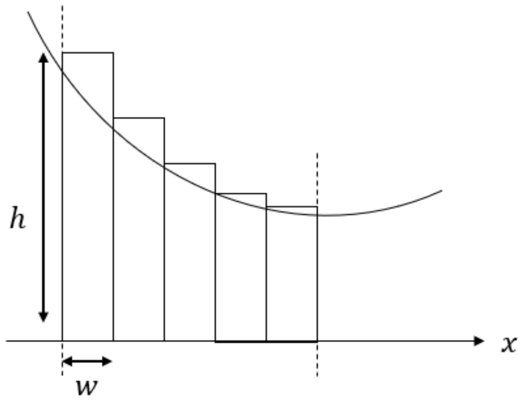
Diagram below shows a function $f(x) = 5^{2x}$.



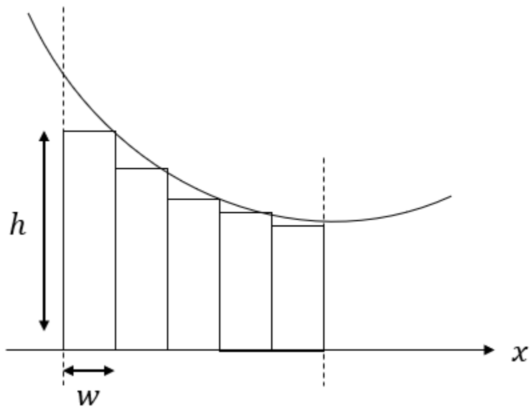
Estimate the shaded area using trapezium rule.

(b) Rectangle method/ midpoint rule

Given a curve with function $f(x)$



Area circumscribed



Area inscribed

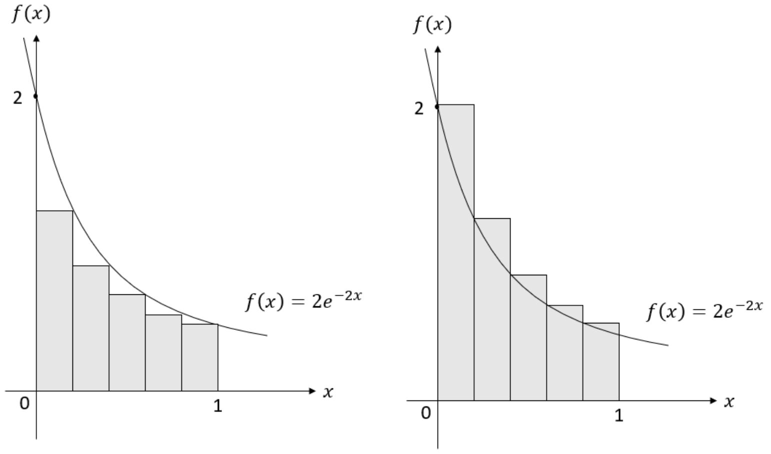
To find each area of strips (rectangles):

$$A = f(x)/\ h \times w$$

Estimating the area of curve,

$$Area \approx \frac{A_{circumscribed} + A_{inscribed}}{2}$$

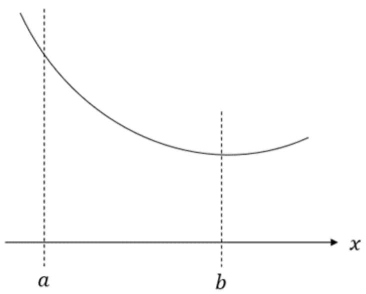
Example:
Diagrams below shows graphs of $f(x) = 2e^{-2x}$ inscribed and circumscribed.



Estimate the area of the region trapped between the curve and x —axis from $x = 0$ to $x = 1$.

(c) Integration (definite integral)

Given a curve with function $f(x)$



$$Area = \int_a^b f(x)\,dx$$

Tips for finding area bounded by two functions:
Use the function above minus the function below.

Example:
Find the area trapped between $f(x) = sin\ x$ and $f(x) = cos\ x$ for the range $0 \leq x \leq \pi$.

6. Fundamental theorem of calculus

(a) Evaluation theorem: Part 2

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Example:
Use the fundamental theorem of calculus to evaluate $\int_0^1 x^2 + e^x \ dx$. Give your answer in terms of e.

(b) Relationship between differentiation and integration: Part 1

Finding derivative using fundamental theorem of calculus

$$\frac{d}{dx}\left[\int_a^x f(t)\right] dt = f(x)$$

Example 1:
Determine $\frac{d}{dx}\ [\int_1^x t^2 + 2]\ dt$.

Example 2:
Determine $\frac{d}{dy}\ [\int_1^y\ 3t^5 + 2t]\ dt$.

Using fundamental theorem & chain rule to calculate derivatives

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\right] dt = f[g(x)] \times g'(x)$$

Example 1:

Find $\frac{d}{dx}\left[\int_1^{x+1} t\right] dt.$

Example 2:

Find $\frac{d}{dx}\left[\int_\pi^{e^x} t^2 + t\right] dt.$

Using fundamental theorem of calculus with two variable limits of integration

Steps:

1. Break the integrals in accordance to $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$
2. Apply $\frac{d}{dx}\left[\int_a^{g(x)} f(t)\right] dt = f[g(x)] \times g'(x)$ and/ or $\frac{d}{dx}\left[\int_a^x f(t)\right] dt = f(x)$ whenever necessary.

Example 1:

Find $f'(x)$ of $f(x) = \int_t^{3t} x^3\, dx.$

Example 2:

Find $\frac{d}{dx}\left[\int_{x+2}^{ln\, 2x} y^2\, dy\right].$

Theorem (iii)

$$\int_b^a \frac{d}{dt}[f(t)]\,dt = f(a) - f(b)$$

Example 1:

Find $\int_2^x \frac{d}{dt}(t^3+1)dt$.

Example 2:

Find $\int_\pi^{x^2} \frac{d}{dt}(2t^2+t)dt$.

7. Additivity and linearity of definite integrals

Summary:

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b k \times f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)dx = 0$$

Example:

Given that $\int_1^7 f(x)\,dx = 3$, evaluate $\int_7^7 2f(x)\,dx$.

Using substitution method

Example:
Given that $f(x)$ is continuous everywhere and that $\int_7^{15} f(x)\,dx = 7$, evaluate $\int_2^{10} f(x+5)\,dx$.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Example 1:
Given that $\int_1^{100} f(x)dx = e^{12}$, evaluate $\int_{100}^{1} f(x)dx$

Example 2:
Given that $\int_{-10}^{-2} f(x)dx = 5$, evaluate $\int_{10}^{2} f(-x)dx$.

$$\int_a^b k \times f(x)dx = k\int_a^b f(x)dx$$

Example 1:
Given that $\int_1^7 f(x)\,dx = 3$, evaluate $\int_1^7 7f(x)\,dx$.

Example 2:
Given that $\int_3^7 f(x)\,dx = 12$, evaluate $\int_3^7 \frac{f(x)}{4}\,dx$.

Example 3:
Given that $\int_5^{12} f(x)\,dx = 7$, evaluate $\int_1^7 2f(x+3)\,dx$.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Example 1:
Given that $\int_{12}^{6} f(x)dx = 100$, evaluate $\int_{12}^{18} f(x)\,dx - \int_6^{18} [f(x)+10]\,dx$.

Example 2:
Given that $\int_1^7 f(x)dx = 13$ and $\int_7^6 f(x)dx = 24$, evaluate $\int_1^7 f(x)dx + \int_6^7 f(x)dx$.

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^b [f(x) \pm c\,]\, dx = \int_a^b f(x)\, dx \pm \int_a^b c\, dx$$

Example 1:
Given that $\int_9^{87} f(x)\, dx = -43$, evaluate $\int_9^{87} [f(x) + 10]\, dx$.

Example 2:
Given that $\int_1^{9} f(x)\, dx = 3.5$, evaluate $\int_1^{9} [f(x) - 10x]\, dx$.

END